Decomposition Method and Upper Bound Density Related to Congruent Saturated Hyperball Packings in Hyperbolic n-Space

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Abstract

We study packings of congruent **hyperballs** (equidistance bodies around hyperplanes) in the n-dimensional hyperbolic space $\mathbb{H}^n$ ($n \geq 4$). For every congruent saturated hyperball packing we give an algorithm that decomposes $\mathbb{H}^n$ into **truncated simplices**. Combined with Miyamoto's theorem, the density upper bound related to truncated simplex cells is attained in a **regular truncated simplex**. In $\mathbb{H}^4$ this upper bound is **≈ 0.75864**, and the density turns out to be **not monotone** in the hyperball height, which **disproves Przeworski's conjecture**.

Keywords: *hyperbolic geometry, hyperball packings, packing density, truncated simplices, Coxeter orthoschemes*

Introduction

In $\mathbb{H}^n$ there are three kinds of "generalized balls": **balls**, **horoballs** and **hyperballs**. A hyperball of height h is the set of points within distance h of a hyperplane, its *base plane*. Packing density in hyperbolic space is measured locally, relative to the cells of a suitable decomposition.

Known density bounds

- $\mathbb{H}^2$: hypercycle packings ≤ 3/π (Vermes, 1979)
- $\mathbb{H}^3$: balls & horoballs ≤ 0.85328 (Böröczky–Florian, 1964)
- $\mathbb{H}^3$: congruent hyperballs ≤ 0.86338, attained in a regular truncated tetrahedron (Szirmai, 2023)
- $\mathbb{H}^4$, $\mathbb{H}^5$: best known hyperball packings come from regular prism tilings (Szirmai)

Research questions

Q1. What is the optimal congruent hyperball packing configuration in $\mathbb{H}^n$ ($n \geq 4$), and how large is its density?

Q2. Does the density grow monotonically with the hyperball height h, as conjectured by A. Przeworski (2013)?

Preliminaries

Projective model. $\mathbb{H}^n$ is the interior of the quadric $\langle x, x\rangle = 0$ in the real projective space $\mathscr{P}^n$, where the Lorentzian form of signature (1, n) is

$$\langle \mathbf{x}, \mathbf{y}\rangle = -x^0y^0 + x^1y^1 + \cdots + x^ny^n$$

Points outside the quadric are *outer points*; each has a *polar hyperplane*. **Hyperball volume.** Extending Bolyai's method, the piece of a hyperball of height h over a polytope $\mathscr{A}_3$ in its base plane (curvature −1) has volume

$$\mathrm{Vol}_4\left(\mathcal{H}_4^{h^+}(\mathcal{A}_3)\right) = \frac{1}{8}\,\mathrm{Vol}_3(\mathcal{A}_3)\left[\frac{2}{3}\sinh 3h + 6\sinh h\right]$$

Local density. For a truncated simplex $\mathscr{S}$ with hyperballs $\mathcal{H}_i^h$ around its truncating faces,

$$\delta(\mathcal{S}(h)) = \frac{\sum_{i=1}^{n+1}\mathrm{Vol}\left(\mathcal{H}_i^h \cap \mathcal{S}\right)}{\mathrm{Vol}(\mathcal{S})}$$

Its supremum over all truncated simplices gives a universal upper bound for congruent saturated hyperball packings.

Research Method

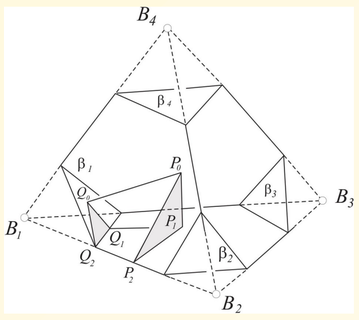
Step 1 — Decomposition algorithm. A packing is *saturated* if no further hyperball can be added; then the base planes satisfy $d(\beta_i, \beta_j) \geq 2h$. Starting from the packing, we build:

1. **Dirichlet–Voronoi cells** from the radical hyperplanes of the hyperspheres; by saturation every vertex is a proper point of $\mathbb{H}^n$.
2. **Compact cells $\mathscr{D}(P)$** at each D–V vertex P: the base planes $\beta_i(P)$ bound a polytope, cut off by the polar planes of its outer vertices.
3. **Cutting** by the polar hyperplane of an outer intersection point of non-adjacent base planes; the number N of such points strictly decreases.
4. **Packing preserved**: the cutting hyperplane does not meet the remaining hyperballs (Lemma 3.1).
5. **Stop at N = 0**: every piece is a truncated simplex carrying a congruent hyperball packing.

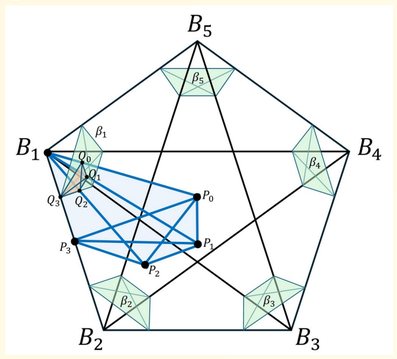
Theorem 1. *The algorithm provides, for each congruent saturated hyperball packing, a decomposition of $\mathbb{H}^n$ into truncated simplices.*

Lemma (reduction). *With Miyamoto's theorem, it suffices to determine the density upper bound in **regular** truncated simplices.*

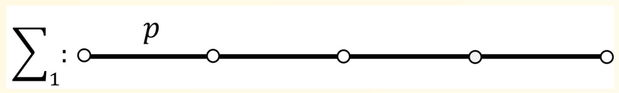
Step 2 — Orthoscheme dissection in $\mathbb{H}^4$. A regular 4-simplex with outer vertices $B_1,\ldots,B_5$, placed symmetrically in the Beltrami–Cayley–Klein model, is truncated by the polar planes $\beta_i = \mathrm{pol}(B_i)$ and cut into **5! = 120** congruent simply truncated orthoschemes $P_0P_1P_2P_3B_1$.



$\mathbb{H}^3$ *analogue (tetrahedron)*



Truncated regular 4-simplex



Coxeter–Schläfli graph Σ_1 of the orthoscheme

Step 3 — Volumes. Schläfli's reduction formula, extended by Kellerhals to complete orthoschemes of degree d in $\mathbb{H}^{2n}$:

$$F_{2n}(\Sigma_d) = \sum_{k=0}^{n}\frac{(-1)^k}{k+1}\binom{2k}{k}\sum_{\sigma} f_{2n-(2k+1)}(\sigma)$$

For n = 2, d = 1 this yields

$$\mathrm{Vol}_4(R_1) = \frac{F_4(\Sigma_1)}{c_4} = \frac{\pi^2}{12}\left(\frac{2}{15} - \frac{2}{3p}\right)$$

The essential angle π/p is tied to the vertex parameter s by $\cos(\pi/p) = \sqrt{(10/(16 - s^2))}$. Outer vertices and disjoint base planes force

$$5.1043 \approx \frac{\pi}{\arccos\sqrt{2/3}} < p < \frac{\pi}{\arccos\sqrt{3/4}} = 6$$

Discussion & Result

The largest admissible hyperball height is half the distance between two base planes:

$$h(p) = \frac{1}{2}\,\mathrm{arccosh}\left(\frac{\cos\frac{2\pi}{p}}{\left|3\cos\frac{2\pi}{p} - 1\right|}\right)$$

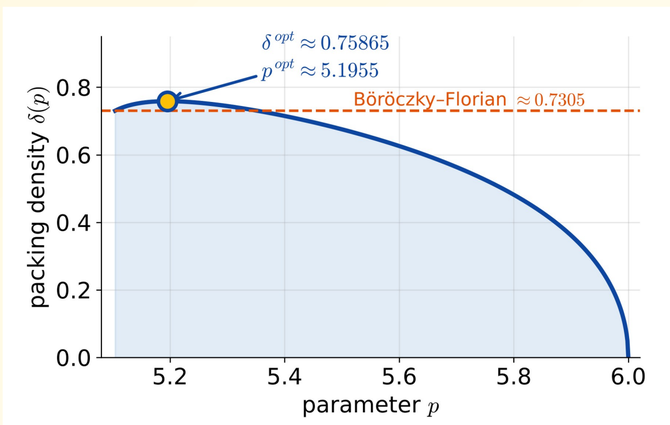
The base $\mathscr{A}_3 = Q_0Q_1Q_2Q_3$ on β_1 is a 3-orthoscheme; Kellerhals' formula with the Lobachevsky function gives

$$\mathrm{Vol}_3(\mathcal{A}_3) = \tfrac{1}{4}\Big\{\mathcal{L}\left(\tfrac{\pi}{p}+\theta\right) - \mathcal{L}\left(\tfrac{\pi}{p}-\theta\right) + \mathcal{L}\left(\tfrac{5\pi}{6}-\theta\right) + \mathcal{L}\left(\tfrac{\pi}{6}-\theta\right) + \mathcal{L}\left(\tfrac{\pi}{3}+\theta\right) - \mathcal{L}\left(\tfrac{\pi}{3}-\theta\right) + 2\mathcal{L}\left(\tfrac{\pi}{2}-\theta\right)\Big\}$$

$$\tan\theta = \frac{\sqrt{1 - 3\sin^2\frac{\pi}{p}}}{\cos\frac{\pi}{p}}, \qquad \mathcal{L}(x) = -\int_0^x \log|2\sin t|\,dt$$

Theorem 2 (local density). *The locally optimal congruent hyperball density in the regular truncated 4-simplex is*

$$\delta(p) = \frac{\frac{1}{8}\,\mathrm{Vol}_3(\mathcal{A}_3(p))\left[\frac{2}{3}\sinh 3h(p) + 6\sinh h(p)\right]}{\frac{\pi^2}{12}\left(\frac{2}{15} - \frac{2}{3p}\right)}$$



Density δ(p) of the locally optimal congruent hyperball packing (5.1043 < p < 6)

upper bound in $\mathbb{H}^4$

0.75865

at $p_{opt} \approx 5.1955$

▲ Exceeds the Böröczky–Florian bound ≈ 0.7305 of balls and horoballs in $\mathbb{H}^4$, which is the limit of δ(p) as p → 5.1043.

▼ δ first rises, then falls as h(p) grows: the density is not monotone in h.

Conclusion

- ✔ A constructive **decomposition algorithm** turns every congruent saturated hyperball packing of $\mathbb{H}^n$ ($n \geq 4$) into truncated simplices.
- ✔ The density upper bound is attained in a **regular truncated simplex** (via Miyamoto's theorem).
- ✔ In $\mathbb{H}^4$: $\delta_{opt} \approx$ **0.7586482** at $p_{opt} \approx 5.1955$.
- ✔ δ is **not an increasing function** of the hyperball height h, hence **Przeworski's conjecture fails** in $\mathbb{H}^4$.
- ✔ The optimal configuration is locally optimal and cannot be extended to all of $\mathbb{H}^4$; the method extends to higher dimensions.

Selected references

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