Keterbatasan Operator Integral pada Ruang Morrey dengan Eksponen Peubah dan Penerapannya dalam Persamaan Diferensial Parsial Eliptik

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Introduction

This research investigates the boundedness of integral operators, regularity properties of elliptic equations, and geometric constants associated with Morrey spaces and their generalizations. The main results include regularity estimates for solutions of elliptic equations in variable Morrey spaces, boundedness of maximal commutators on a closed subspace of Orlicz-Morrey spaces, and non-density phenomena in discrete Morrey spaces together with applications to fractional operators. We also establish the boundedness of the Laplace transform on Morrey-type spaces and Morrey-Lorentz spaces. Furthermore, we introduce the $n-$th Zbáganu constant on Banach spaces.

Main Research

1. Regularity Of Solutions To Elliptic Equations In Variable Morrey Spaces

The following theorems establish the regularity properties and a priori estimates for solutions to non-divergence form elliptic equations on variable Morrey spaces.

Theorem 1.1 *Let $p(\cdot) \in \mathrm{LH}(\Omega)$, $1 < p_- \le p_+ < \infty$, and $\lambda(\cdot) : \Omega \to [0, n]$. Given $\Omega' \Subset \Omega'' \Subset \Omega$. Then there exists a positive constant C, independent of u, such that*

$$\|D^2u\|_{M^{\lambda(\cdot)}_{p(\cdot)}(\Omega')} \le C\left(\|u\|_{M^{\lambda(\cdot)}_{p(\cdot)}(\Omega'')} + \|L_0u\|_{M^{\lambda(\cdot)}_{p(\cdot)}(\Omega'')}\right)$$

This result can be extended to the following boundary estimates.

Theorem 1.2 *Let $p(\cdot) \in \mathrm{LH}(\Omega)$, $1 < p_- \le p_+ < \infty$, and $\lambda(\cdot) : \Omega \to [0, n]$. Let $b_1, b_2, \dots b_n, c \in L^n(\Omega) \cap M^1_n(\Omega)$ with $c(x) \neq 0$ for a.e. $x \in \Omega$. Assume that $f \in M^{\lambda(\cdot)}_{p(\cdot)}(\Omega)$ and $u \in W^2M^{\lambda(\cdot)}_{p(\cdot)}(\Omega) \cap W^{1,p(\cdot)}_0(\Omega)$ are such that $Lu(x) = f(x)$ for a.e. $x \in \Omega$. Then $D^2u \in M^{\lambda(\cdot)}_{p(\cdot)}(\Omega)$ and there exists a positive constant, independent of u and f, such that*

$$\|D^2u\|_{M^{\lambda(\cdot)}_{p(\cdot)}(\Omega)} \le C\left(\|u\|_{M^{\lambda(\cdot)}_{p(\cdot)}(\Omega)} + \|f\|_{M^{\lambda(\cdot)}_{p(\cdot)}(\Omega)}\right).$$

By using this boundary estimate, we prove the following result.

Theorem 1.3 *Keep the same assumptions as in Theorem 1.2. Then, for every $f \in M^{\lambda(\cdot)}_{p(\cdot)}(\Omega)$, there exists a unique solution of the Dircihlet problem*

$$\begin{cases} Lu(x) = f(x) \text{ for a.e. } x \in \Omega, \\ u \in W^2M^{\lambda(\cdot)}_{p(\cdot)}(\Omega) \cap W^{1,p(\cdot)}_0(\Omega). \end{cases}$$

Morever, there exists a positive constant C, independent of u and f, such that

$$\|D^2u\|_{M^{\lambda(\cdot)}_{p(\cdot)}} \le C\|f\|_{M^{\lambda(\cdot)}_{p(\cdot)}}.$$

2. Maximal commutators on a closed subspace of Orlicz-Morrey spaces

In this research, we prove that the maximal commutator M_b and the commutator of the Hardy–Littlewood maximal operator $[b, M]$ are bounded on a vanishing subspace of Orlicz-Morrey space $V^{(*)}\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$. Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The maximal commutator M_b is an operator that maps f to M_bf, where

$$M_bf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(x) - b(y)||f(y)|\, dy.$$

The commutator of the Hardy-Littlewood maximal operator, denoted by, $[b, M]$, maps each $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ to $[b, M]f := b \cdot Mf - M(bf)$. Assume that b belongs bounded mean oscillation class and under suitable assumptions on the functions Φ and φ, we have the following results.

Theorem 2.1 *Let $b \in BMO(\mathbb{R}^n)$, Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$ and $\varphi \in \mathcal{G}_\Phi$ satisfy the condition*

$$\sup_{r<t<\infty} \left(1 + \ln\frac{t}{r}\right)\Phi^{-1}\big(t^{-n}\big)\varphi(t) \lesssim \Phi^{-1}\big(r^{-n}\big)\varphi(r).$$

Let also

$$\lim_{t\to\infty} \varphi(t)\Phi^{-1}(t^{-n}) \ln t = 0.$$

Then we have the following results:

1. The maximal commutator M_b is bounded on $V^{(*)}\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$;

2. The commutator of the Hardy-Littlewood maximal operator $[b, M]$ is bounded on $V^{(*)}\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.

3. Diversity in discrete Morrey spaces and its application to fractional operators and their commutators

In this research, we obtain an alternative proof of the uniqueness parameters of discrete Morrey spaces and a necessary condition for the boundedness of the fractional maximal operator M_α, the fractional integral operator I_α, and the commutators of I_α on discrete Morrey spaces.

Theorem 3.1 *Let $1 \le q_0 \le p_0 < \infty$ and $1 \le q_1 \le p_1 < \infty$. Then $m^{p_0}_{q_0}(\mathbb{Z}^n) = m^{p_1}_{q_1}(\mathbb{Z}^n)$ if and only if $p_0 = p_1$ and $q_0 = q_1$.*

Theorem 3.2 *Let $0 \le \alpha < n$, $1 < q \le p < \infty$, $1 < t \le s < \infty$, and $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$. If M_α or I_α (for $0 < \alpha < n$) is bounded from $m^p_q(\mathbb{Z}^n)$ to $m^s_t(\mathbb{Z}^n)$, then necessarily $\frac{q}{p} \ge \frac{t}{s}$.*

Theorem 3.3 *Let $0 < \alpha < n$, $1 < q \le p < \infty$, $1 < t \le s < \infty$, and $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$. Suppose that $b \in \mathrm{BMO}(\mathbb{Z}^n)$. Then, the boundedness of $[b, I_\alpha]$ from $m^p_q(\mathbb{Z}^n)$ to $m^s_t(\mathbb{Z}^n)$ implies $\frac{q}{p} \ge \frac{t}{s}$.*

In addition, we obtain the nondensity results for inclusion between discrete Morrey spaces.

Theorem 3.4 *Let $1 \le q < p < \infty$. Then $\bigcup_{r\in(q,p]} m^p_r(\mathbb{Z}^n)$ is not dense in $m^p_q(\mathbb{Z}^n)$, and for any $u > 1$, $m^p_q(\mathbb{Z}^n)$ is not dense in $m^{pu}_{qu}(\mathbb{Z}^n)$.*

Additional Research

4. Boundedness of the Laplace Transform on Morrey-type Spaces

The following results establish the boundedness of the Laplace transform $\mathcal{L}$ on weighted Morrey-type and Morrey–Lorentz spaces. For Morrey-type spaces, the results are stated in the following theorems:

Theorem 4.1 *Let $w_p(x) = x^{p-2}$. Then we have*

1. $\mathcal{L} : {}^cL^{p,\lambda,w_p}_{0,\theta} \to L^{p,\lambda}_{0,\theta}$ is bounded with bound $\Gamma(\frac{1-\lambda}{p})$ for $0 \le \lambda < 1$;

2. $\mathcal{L} : L^{p,\lambda,w_p}_{0,\theta} \to {}^cL^{p,\lambda}_{0,\theta}$ is bounded with bound $\Gamma(\frac{1+\lambda}{p})$ for $\lambda \ge 0$.

Theorem 4.2 *Let $1 < p < \infty$ and $w_p(x) := x^{p-2}$. Then we have*

1. $\mathcal{L} : {}^cL^{p,\lambda}_{0,\theta} \to L^{p,\lambda,w_p}_{0,\theta}$ is bounded with bound $\Gamma(\frac{1}{p'} - \frac{\lambda}{p})$ for $0 \le \lambda < p - 1$;

2. $\mathcal{L} : L^{p,\lambda}_{0,\theta} \to {}^cL^{p,\lambda,w_p}_{0,\theta}$ is bounded with bound $\Gamma(\frac{1}{p'} + \frac{\lambda}{p})$ for $\lambda \ge 0$.

More generally, the result extends to weighted parameters as follows:

Theorem 4.3 *Let $1 \le \theta \le \infty$. Write $w_\gamma(x) := x^{\gamma p}$ and $w_\delta(x) := x^{\delta p}$, where $\gamma < \frac{1}{p'}$ and $\delta = \frac{p-2}{p} - \gamma$. Then we have*

1. $\mathcal{L} : {}^cL^{p,\lambda,w_\gamma}_{0,\theta} \to L^{p,\lambda,w_\delta}_{0,\theta}$ is bounded with bound $\Gamma(\frac{1}{p'} - \gamma - \frac{\lambda}{p})$ for $0 \le \lambda < (1-\gamma)p - 1$;

2. $\mathcal{L} : L^{p,\lambda,w_\gamma}_{0,\theta} \to {}^cL^{p,\lambda,w_\delta}_{0,\theta}$ is bounded with bound $\Gamma(\frac{1}{p'} - \gamma + \frac{\lambda}{p})$ for $\lambda \ge 0$.

Extending the investigation to Morrey-Lorentz spaces, the main boundedness result is given by the following theorem:

Theorem 4.4 *Let $1 \le p, q < \infty$ and $\gamma = 1 - \frac{2}{q}$. Write $w_\gamma(x) := x^{\gamma p}$. Then we have*

1. $\mathcal{L} : {}^cL^{p,q,\lambda,w_\gamma}_0 \to L^{p,q,\lambda}_0$ is bounded with bound $A_{p,q} = \frac{pq}{p-\lambda q}$ for $0 \le \lambda < \frac{p}{q}$;

2. $\mathcal{L} : L^{p,q,\lambda,w_\gamma}_0 \to {}^cL^{p,q,\lambda}_0$ is bounded with bound $B_{p,q} = \frac{pq}{p+\lambda q}$ for $\lambda \ge 0$.

5. The n-th Zbáganu constant

We present bounds and exact values for the n-th Zbáganu constant. We also establish its relationship to the n-th James and n-th Von Neumann–Jordan constants, as stated in the following theorem.

Theorem 5.1 *For any Banach space X, we have*

$$[C^{(n)}_J(X)]^2 \le nC^{(n)}_Z(X) \le nC^{(n)}_{NJ}(X),$$

for every $n \ge 2$.

As a direct consequence regarding the bounds and the property for Hilbert spaces, the following corollary is obtained:

Corollary 5.2 *For every $n \ge 2$, we have $1 \le C^{(n)}_Z(X) \le n$. Moreover, if X is a Hilbert space, then $C^{(n)}_Z(X) = 1$.*

In classical spaces, the value of the n-th Zbáganu constant is determined by the following theorem:

Theorem 5.3 *For $X = L^p$ and $X = \ell^p$, where $1 \le p \le 2$, we have $C^{(n)}_Z(X) = n^{2/p} - 1$.*

Furthermore, by the construction of functions and sequences for Morrey spaces and discrete Morrey spaces, we established the following result:

Theorem 5.4 *For $1 \le p < q < \infty$ and $n \ge 2$, we have*

$$C^{(n)}_Z(\mathcal{M}^p_q) = C^{(n)}_Z(\ell^p_q) = n.$$

Dissemination

1. H. Gunawan, F. Hanafi, and N. R. P. Widjaja, “Some norm estimates for the Laplace transform”, The Fourth International Conference on Mathematical Analysis and Its Applications (IConMAA), Universitas Hassanudin Makassar, Makassar, 23-24 Juli 2026.
2. D. I. Hakim *et al.*, “Resent advances in Kantorovich-type operators”, The Fourth International Conference on Mathematical Analysis and Its Applications (IConMAA), Universitas Hassanudin Makassar, Makassar, 23-24 Juli 2026.
3. Y. Ramadana and H. Gunawan, “Variable exponent discrete Lebesgue spaces and the discrete fractional integral with bivariate quadratic form”, The Fourth International Conference on Mathematical Analysis and Its Applications (IConMAA), Universitas Hassanudin Makassar, Makassar, 23-24 Juli 2026.
4. F. Hanafi, “On the boundedness of integral transforms asssociated with Bessel and Struve functions in Morrey-type spaces”, The Fourth International Conference on Mathematical Analysis and Its Applications (IConMAA), Universitas Hassanudin Makassar, Makassar, 23-24 Juli 2026.

Publication

1. R. Roohi Seraji and D. I. Hakim, “Regularity of solutions to elliptic equations in variable Morrey spaces”, draft.
2. F. Deringoz and D. I. Hakim, “Maximal commutators on a closed subspace of Orlicz–Morrey spaces”, published in Transactions of the National Academy of Sciences of Azerbaijan, Series of Physical-Technical and Mathematical Sciences: Issue Mathematics 2026.
3. K. Hayashi, K. Hiramatsu, K. Ikeya, Y. Kawakami, A. Kobayashi, R. Masuda, S. Mochizuki, R. Murai, T. Murai, T. Nonaka, M. Onigata, Y. Sawano, D. Takesako, and D. I. Hakim, “Diversity in discrete Morrey spaces and its application to fractional operators and their commutators”, to appear in Annals of Functional Analysis.
4. F. Hanafi and H. Gunawan, “Boundedness of the Laplace Transform on Morrey-type Spaces”, under review.
5. Ifronika, H. Gunawan, and J. Lindiarni, “The n-th Zbáganu Constant”, submitted.

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