Pengembangan Wide-Array of Nonlinear Dynamics Approximation (WyNDA) untuk Identifikasi Model Persamaan Diferensial Berbasis Data

PPMI FMIPA Kerjasama International

S.R. Pudjaprasetya[1], A. Hasan[2], A. Widyotriatmi[3], R. Kusdiantara[1] , Iryanto[1]

[1]Fakultas Matematika dan Ilmu Pengetahuan Alam – ITB, [2]Norwegian University of Science and Technology, [3]Fakultas Teknologi Industri - ITB

Abstract

This work extends the Wide-Array of Nonlinear Dynamics Approximation (WyNDA) from ordinary differential equations to partial differential equations (PDEs). The proposed semi-discrete framework employs an adaptive observer to jointly estimate system states and PDE parameters from measurement data. A candidate library incorporating spatial derivatives is constructed, while a redundancy–relevance pruning strategy eliminates inactive and linearly dependent terms. The framework is evaluated on a range of benchmark equations, including heat, advection–diffusion, Burgers, Allen–Cahn, Korteweg–de Vries, Kuramoto–Sivashinsky, and shallow water equations. Results demonstrate its ability to accurately identify governing terms, highlighting the importance of compact, physics-aligned approximation bases. Additional experiments with spatially sparse measurements demonstrate the framework's effectiveness even with limited observational information. These findings establish WyNDA as an interpretable, observer-based framework for data-driven PDE discovery.

Keyword: Data-driven discovery, Data-driven PDEs discovery, WyNDA, Adaptive observer, Joint state and parameter estimation.

Introduction

Physical laws are traditionally derived from first principles, such as conservation of mass, momentum, and energy. However, deriving governing equations becomes challenging for complex systems or domains with limited theoretical understanding, motivating data-driven approaches that identify governing dynamics directly from observational or experimental data.

Existing data-driven methods include sparse regression, such as Sparse Identification of Nonlinear Dynamics (SINDy), machine learning approaches, such as Physics-Informed Neural Networks (PINNs), and observer-based methods, such as Wide-Array of Nonlinear Dynamics Approximation (WyNDA). This work focuses on WyNDA, which combines library-based modeling with adaptive joint state and parameter estimation through filtering. However, its applications have primarily focused on ordinary differential equations (ODEs), with limited extensions to partial differential equations (PDEs). To address this gap, we extend WyNDA to PDEs through a semi-discrete formulation and incorporate a redundancy–relevance pruning strategy to eliminate inactive or linearly dependent terms during estimation.

The proposed framework is evaluated on linear and nonlinear benchmark PDEs, coupled shallow water equations, and a nonlinear dam-break problem. Additional experiments with spatially sparse measurements examine its performance under limited observational information.

Research Method

We consider PDEs of the general form

$$u_t + F(u, u_x, u_{xx}, \dots, x, t; \theta) = p(x) \qquad (1)$$

subject to $u(x,0) = u_0(x)$ and appropriate boundary conditions.

Let u_j^n denote the data at $x_j = j\Delta x$, and $t_n = n\Delta t$. Define $\mathbf{u}(n) = [u_1^n, \dots, u_N^n]^T \in R^N$ and $\mathbf{p} = [p(x_1), \dots, p(x_N)]^T \in R^N$. Applying the forward Euler scheme in time gives:

$$\boldsymbol{u}(n+1) = \boldsymbol{u}(n) + \Delta t\big[-\boldsymbol{F}\big(\boldsymbol{u}(n)\big)\boldsymbol{\theta}\big] + \Delta t\boldsymbol{p}. \qquad (2)$$

Approximating the dynamics using r candidate basis functions yields:

$$\boldsymbol{u}(n+1) = \boldsymbol{u}(n) + \boldsymbol{\Psi}[\boldsymbol{u}(\boldsymbol{n})]\boldsymbol{\vartheta} + \Delta t\boldsymbol{p}, \qquad (3)$$

where $\boldsymbol{\Psi}[\boldsymbol{u}(n)] \in R^{N\times r}$ is the the basis-function regressor, $\boldsymbol{\vartheta} \in \boldsymbol{R^r}$ contains the discrete-time coefficients to be estimated, and $\boldsymbol{\theta} \approx \frac{\boldsymbol{\vartheta}}{\Delta t}$.

We employ the adaptive observer to jointly estimate the states and parameters of the approximation model (3) .The observer updates are given by:

$$\hat{\mathbf{u}}(n|n) = \hat{\mathbf{u}}(n|n-1) + [\mathbf{K}_u(n) + \boldsymbol{\Gamma}(n|n)\,\mathbf{K}_\theta(n)]\big(\mathbf{u}(n) - \hat{\mathbf{u}}(n|n-1)\big),$$

$$\hat{\boldsymbol{\vartheta}}(n|n) = \hat{\boldsymbol{\vartheta}}(n|n-1) - \mathbf{K}_\theta(n)\big(\mathbf{u}(n) - \hat{\mathbf{u}}(n|n-1)\big),$$

Observer gains and sensitivity matrix are computed as

$$\mathbf{K}_u(n) = \mathbf{P}_u(n|n-1)[\mathbf{P}_u(n|n-1) + \mathbf{R}_u(n)]^{-1},$$

$$\boldsymbol{\Omega}(n) = \boldsymbol{\Gamma}(n|n-1)\,\mathbf{P}_\vartheta(n|n-1)\,\boldsymbol{\Gamma}^{\mathrm{T}}(n|n-1) + \mathbf{R}_\vartheta(n),$$

$$\mathbf{K}_\theta(n) = \mathbf{P}_\vartheta(n|n-1)\,\boldsymbol{\Gamma}^{\mathrm{T}}(n|n-1)\,\boldsymbol{\Omega}(n)^{-1},$$

$$\boldsymbol{\Gamma}(n|n) = [\mathbf{I} - \mathbf{K}_u(n)]\boldsymbol{\Gamma}(n|n-1).$$

Finally, prediction updates are given by:

$$\hat{\mathbf{u}}(n+1|n) = \hat{\mathbf{u}}(n|n) + \boldsymbol{\Psi}\big(\mathbf{u}(n)\big)\,\hat{\boldsymbol{\vartheta}}(n|n) + \Delta t\mathbf{p},$$

$$\hat{\boldsymbol{\vartheta}}(n+1|n) = \hat{\boldsymbol{\vartheta}}(n|n),$$

$$\mathbf{P}_u(n+1|n) = \lambda_u^{-1}[\mathbf{I} - \mathbf{K}_u(n)]\mathbf{P}_u(n|n-1),$$

$$\mathbf{P}_\vartheta(n+1|n) = \lambda_\vartheta^{-1}[\mathbf{I} - \mathbf{K}_\theta(n)\boldsymbol{\Gamma}(n|n-1)]\mathbf{P}_\vartheta(n|n-1),$$

$$\boldsymbol{\Gamma}(n+1|n) = \boldsymbol{\Gamma}(n|n) - \boldsymbol{\Psi}\big(\mathbf{u}(n)\big).$$

Here, $\lambda_u, \lambda_\vartheta \in (0,1)$ are forgetting factors controlling state and parameter adaptation. The parameter prediction assumes time-invariant PDE coefficients.

To obtain a parsimonious model with the correct governing terms, we employ a redundancy–relevance pruning strategy based on basis-function correlations and parameter magnitude–confidence criteria.

Discussion & Result

For the Burgers equation, WyNDA correctly identifies advection and viscosity terms, suppressing spurious terms, with reconstruction errors below 0.06%. It also correctly identifies zero viscosity in the inviscid case. For the dry-bed dam-break problem, the relative errors are 0.2218% for wave elevation and 5.2978% for velocity, with faster convergence for the equation requiring fewer parameters. The Allen–Cahn equation is accurately identified in both configurations, with reconstruction errors of 0.1702%. Finally, comparisons on KdV and KS benchmarks show lower mean parameter errors for WyNDA than PDE-FIND. Overall, WyNDA successfully identifies governing terms across nonlinear and coupled PDEs, while convergence depends on model complexity and parameter coupling.

True PDE	WyNDA with Prunning
Burgers Equation $u_t + \gamma u u_x - \nu u_{xx} = 0$ $\gamma = 1 \quad \nu = 0.1$ $\gamma = 1 \quad \nu = 0.01$ $\gamma = 1 \quad \nu = 0.001$ $\gamma = 1 \quad \nu = 0$	ϵ_{rel}: 0.044%, ϵ_{rel}: 0.0534% ϵ_{rel}: 0.0565%, ϵ_{rel}: 0.0553% $\gamma = 1 \quad \nu = 0.1001$ $\gamma = 1 \quad \nu = 0.0101$ $\gamma = 0.9999 \quad \nu = 0.0011$ $\gamma = 1 \quad \nu = 0$
SWEs – Dry bed dambreak $\eta_t + (hu)_x = 0$ $u_t + uu_x + 9.81\eta_x = 0$ $\eta(x,0) = 1, x \le 3,\ x \in [0,6]$ $\eta(x,0) = 0, x > 3$ $u(x,0) = 0, t \in [0,0.9]$	$\eta_t + (hu)_x = 0$ $u_t + 0.9994uu_x$ $+9.808\eta_x = 0$ $\epsilon_{rel\eta} = 0.2218\%$ $\epsilon_{relu} = 5.2978\%$
Allen – Cahn Equation $u_t = 5u - 5u^3 + 0.0025u_{xx}$ $u_t = 5u - 5u^3 + 0.0025u_{xx}$ $+0.01\sin(x)$ $u(x,0) = 0.1\cos(x)$	$u_t = 5.0003u - 5.0019u^3$ $+0.0025u_{xx}$ ϵ_{rel}: 0.1702% $u_t = 5.0003u - 5.0019u^3$ $+0.0025u_{xx} + 0.01\sin(x)$ ϵ_{rel}: 0.1702%
KdV Equation (Rudy et al., 2017) $u_t + 6uu_x + u_{xxx} = 0$ $x \in [-30, 30],\ N = 512$	ϵ_{rel}: 2.7679% $u_t + 6.0028uu_x + u_{xxx} = 0$ $t \in [0, 20],\ M = 201$
KS Equation (Rudy et al., 2017) $u_t + uu_x + u_{xx} + u_{xxxx} = 0$ $x \in [0, 100],\ N = 1024$	ϵ_{rel}: 7.0718% $u_t + 0.9999uu_x + 0.9962u_{xx}$ $+1.0007u_{xxxx} = 0$ $t \in [0, 100],\ M = 251$

As shown in the following table, partial measurements yield reconstructions comparable to full observations, with only minor differences in parameter estimates and reconstruction errors while preserving the dominant governing dynamics.

True PDE	Recovered Model
Homogeneous Heat Equation	
$u_t = 0.04u_{xx}$ $u(x,0) = 2\sin(\pi x) + \sin(2\pi x)$	$u_t = 0.0401u_{xx}$ $\epsilon_c = \epsilon_m = 10^{-5}$ $\epsilon_{rel} = 0.1418,\ m = 4$
Non Homogeneous Heat Equation	
$u_t - 0.001u_{xx} = 0.002$ $u(x,0) = \sin(\pi x) + x(1-x)$	$u_t - 0.001u_{xx} = 0.002$ $\epsilon_r = 0.98,\ \epsilon_c = 4 \times \epsilon_m = 10^{-7}$ $\epsilon_{rel} = 0.0019,\ m = 4$
Advection – Diffusion Equation	
$u_t = 0.5u_x + 0.01u_{xx} + 2$ $u(x,0) = \sin(0.5\pi x) + \sin(\pi x)$	$u_t = 0.5u_x + 0.01u_{xx} + 1.9999$ $\epsilon_r = 0.97,\ \epsilon_c = \epsilon_m = 10^{-6}\ \epsilon_r = 0.98$ $\epsilon_{rel} = 0.2426,\ m = 2$
Burgers Equation	
$u_t + \gamma u u_x - \nu u_{xx} = 0$ $\gamma = 1 \quad \nu = 0.1$ $\gamma = 1 \quad \nu = 0.01$ $\gamma = 1 \quad \nu = 0.001$ $\gamma = 1 \quad \nu = 0$	$\epsilon_c = \epsilon_m = 3 \times 10^{-7},\ m = 6$ $\gamma = 1 \quad \nu = 0.1001\ \epsilon_{rel} = 0.0441$ $\gamma = 1 \quad \nu = 0.0101\ \epsilon_{rel} = 0.0538$ $\gamma = 0.9998 \quad \nu = 0.0011\ \epsilon_{rel} = 0.0569$ $\gamma = 0.9997 \quad \nu = 0\ \epsilon_{rel} = 0.0583$

Conclusion

This study extends WyNDA from ordinary to partial differential equations through a semi-discrete formulation and adaptive observer, enabling simultaneous state reconstruction and parameter estimation. A redundancy–relevance pruning strategy is incorporated to eliminate inactive and linearly dependent basis functions, while exponential convergence is established under standard persistence-of-excitation assumptions. Numerical validation across linear, nonlinear, and coupled PDEs demonstrates the method's ability to identify governing terms and reconstruct system states. Comparisons with PDE-FIND on KdV and KS benchmarks show comparable or lower parameter estimation errors. Experiments with spatially sparse measurements further demonstrate its effectiveness with limited observational information. The results highlight the importance of compact, physics-aligned libraries, particularly as system complexity increases. They also reveal intrinsic identifiability limitations under rank-deficient conditions, which can be addressed through multimode excitation or forcing. Future work will investigate noisy measurements, more general boundary conditions, higher-dimensional PDEs, and physics-informed regularization.

Outputs:

1. Iryanto, A. Hasan, S.R. Pudjaprasetya, Sensitivity-Based Dual Kalman Filter for Joint State–Parameter Estimation in LTV Systems, accepted for publication at System & Control Letters
2. Iryanto, A. Hasan, A. Widyotriatmo, S.R. Pudjaprasetya, Data-Driven Discovery of PDEs Based on Wide-Array of Nonlinear Dynamics Approximations, https://ieeexplore.ieee.org/abstract/document/11702185
3. Iryanto, A. Hasan, A. Widyotriatmo, S.R. Pudjaprasetya, Data-Driven Modeling of Nonlinear Systems Using a Dual Kalman Filter, https://ieeexplore.ieee.org/abstract/document/11702247
4. P. V. Swastika, K. Dharmawan, I.P.W. Gautama, S. R. Pudjaprasetya, L.H. Wiryanto, V. Gopal, Numerical Modeling of Wave Attenuation at Uluwatu Beach Bali by Submerged Breakwaters Using a New Finite Element Method, MethodsX, https://www.sciencedirect.com/science/article/pii/S2215016126003651