Total Edge Irregularity Strength of Cubic Graphs

Aleams Barra[1], Suhadi Wido Saputro[2], Hajime Tanaka[3]

[1] Algebra Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung, Indonesia
[2] Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung, Indonesia
[3] Research Center for Pure and Applied Mathematics, Graduate School of Information Sciences, Tohoku University, Sendai, Japan

Overview

We construct optimal total edge irregular labelings for two classes of cubic graphs: Hamiltonian graphs and graphs with a perfect matching.

Assign labels in $\{1, \ldots, s\}$ to vertices and edges so that every edge has a different weight:

$$w(uv) = f(u) + f(uv) + f(v).$$

For n vertices, a cubic graph has $\frac{3n}{2}$ edges. Counting available weights gives

$$\mathrm{tes}(G) \geq \left\lceil \frac{|E(G)| + 2}{3} \right\rceil = \frac{n}{2} + 1.$$

Both constructions attain this bound. The perfect-matching result covers a broader class, since the chords of a Hamiltonian cycle form a perfect matching.

Shared labeling method

Set $s = \frac{n}{2} + 1$. Initially label the vertices in A by 1 and those in B by s. Define

$$a = |E(A)|, \quad b = |E(A, B)|, \quad c = |E(B)|.$$

The three edge classes have endpoint sums 2, $s + 1$, and $2s$. Their permitted weights lie in

$$[3, s + 2], \quad [s + 2, 2s + 1], \quad [2s + 1, 3s].$$

ADMISSIBILITY CRITERION

$$a, b, c \leq s,$$
$$a + b,\ b + c \leq 2s - 1,$$
$$a + b + c \leq 3s - 2.$$

Choose different permitted weights w, then assign each edge the label w minus its endpoint sum.

Optimal strength

For either class of finite simple cubic graphs,

$$\boxed{\mathrm{tes}(G) = \frac{n}{2} + 1.}$$

Both approaches bring the internal edge counts into $\{s - 2, s - 1, s\}$. The only exceptional three-class distribution is

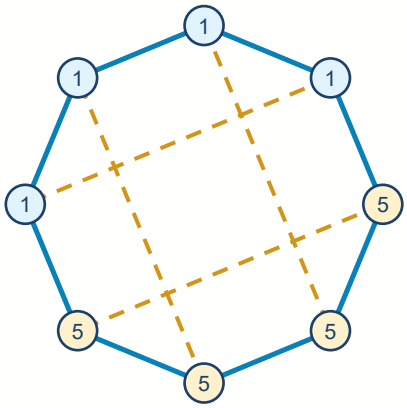
$$(a, b, c) = (s - 2, s + 1, s - 2).$$

Change one vertex labeled s with at least two neighbors labeled 1 to label $s - 1$. An explicit edge-label assignment completes the construction.

Hamiltonian graphs

A Hamiltonian cycle visits every vertex exactly once. Its complementary edges form a perfect matching of chords.

The example below has $n = 8$ and $s = 5$. Blue cycle edges and gold chords display the cubic structure.



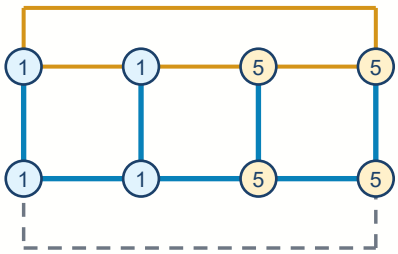
Hamiltonian cycle C (blue), chords (gold).
Vertex labels: four 1s and four 5s.

PATH WITH ATTACHED LEAVES

Choose a chord with a shortest minor cycle arc. The major arc supplies a long path. Attach its endpoint edges and the chords from the minor-arc interior to obtain a spanning tree T.

$$E(G) = E(T) \dot{\cup} E(C_0) \dot{\cup} M_0.$$

Use a path prefix and its attached leaves to form the initial parts. The internal non-tree edges guide local relabeling along the path.



Same graph: T (blue), C_0 (gold), M_0 (gray).

With $m = \frac{n}{2}$ and q internal non-tree edges per part, the initial counts are

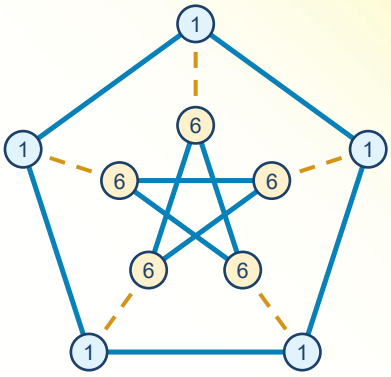
$$a = c = m - 1 + q,$$
$$b = m + 2 - 2q.$$

Each effective flip reduces the relevant internal count by 3. In this illustration $q = 1$, so $(a, b, c) = (4, 4, 4)$.

Perfect matching

A perfect matching M covers each vertex once. Deleting M leaves a spanning 2-factor: disjoint cycles.

The Petersen graph illustrates this approach with $n = 10$ and $s = 6$. Gold spokes form M.

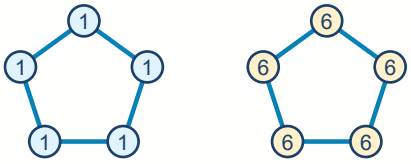


Petersen graph: M is the gold matching.
Vertex labels: five 1s and five 6s.

CYCLE-BASED PARTITION

Order the cycles by length. Label whole cycles and then, if necessary, one consecutive cycle segment by 1 until exactly $\frac{n}{2}$ vertices are labeled. Label the rest by s. At most one cycle is split.

After deleting M: two disjoint 5-cycles



LOCAL FLIPS ON BOTH SIDES

Flip a vertex only when all three neighbors retain its old label. An effective flip reduces one internal count by 3. Equal numbers of opposite flips preserve the balanced partition and give

$$(a, b, c) = (s - i, s - 3 + 2i, s - i), \quad i \in \{0, 1, 2\}.$$

Here both cycles are monochromatic. The counts are $(5, 5, 5)$, so no flips are needed.

Reference

M. Bača, S. Jendrol', M. Miller, J. Ryan. On irregular total labellings. Discrete Mathematics 307 (2007), 1378–1388.

Further Result

We have extended the method to all finite simple cubic graphs, connected or disconnected.