Restoring Finite-Dimensionality of Universal C*-Algebras Generated by Nilpotent Partial Isometries

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Abstract

We study universal C*-algebras generated by a single partial isometry subject to nilpotency and polynomial relations. We show that the universal C*-algebra generated by a nilpotent partial isometry of order at most n is nonunital for every n ≥ 3, and hence infinite-dimensional, whereas the order-two case is isomorphic to a full matrix algebra.

We then impose an explicit polynomial relation in addition to the partial-isometry relation. This forces the generator to be a nilpotent power partial isometry, and the resulting universal C*-algebra is isomorphic to a full matrix algebra.

Finally, if only the power-partial-isometry and nilpotency conditions are retained, the universal C*-algebra is finite-dimensional and isomorphic to a direct sum of full matrix algebras of consecutive orders.

Keywords: universal C*-algebra, partial isometry, power partial isometry, nilpotent operator, matrix algebra.

Introduction

Coburn's theorem identifies the universal C*-algebra of an isometry ($v^*v = 1$) with the Toeplitz algebra, an infinite-dimensional object. Passing to a partial isometry ($vv^*v = v$), Brenken and Niu showed the universal algebra becomes nonunital and still infinite-dimensional.

An element v is a **power partial isometry** (PPI) if every power v^k is a partial isometry. By the Halmos–Wallen decomposition a PPI is a direct sum of a unitary, unilateral shifts, backward shifts and truncated shifts. Roch identified the universal C*-algebra of a PPI with a modified finite-sections algebra: unital, but infinite-dimensional.

Question. *Can nilpotency force finite-dimensionality, and how does it interact with universal constructions defined by partial-isometry relations?*

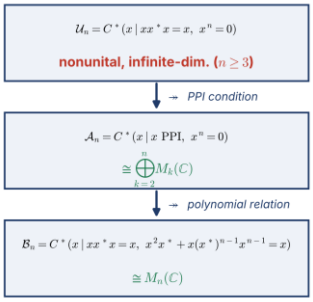
For n = 2 the answer is classical: $C^*(x \mid xx^*x = x, x^2 = 0) \cong M_2(\mathbb{C})$, via $x \mapsto J_2$. We study what happens for higher orders.

Research Method

We consider three universal C*-algebras, for n ≥ 2:

- $U_n = C^*(x \mid xx^*x = x,\ x^n = 0)$;
- $A_n = C^*(x \mid x \text{ is a PPI},\ x^n = 0)$;
- $B_n = C^*(x \mid xx^*x = x,\ x^2x^* + x(x^*)^{n-1}x^{n-1} = x)$.

The polynomial relation extends the order-two case: for a partial isometry, $v^2 = 0 \Leftrightarrow v^2v^* + vv^*v = v$. Universality yields canonical surjections $U_n \to A_n \to B_n$ (Picture 1).



Picture 1. The three universal C-algebras*

Representation technique (U_n).
Represent x on $\bigoplus_k \mathbb{C}^4$ by $T = \bigoplus_k A_k$ (Picture 2, left). Each block satisfies $A_kA_k^*A_k = A_k$, $A_k^3 = 0$, and I_4 is a $*$-polynomial without constant term in A_k, A_k^*. Since $A_k \to J_3 \oplus 0$ in norm, a would-be unit cannot be approximated uniformly.

Spanning sets (B_n, A_n).
Using the identity relation, every $*$-monomial in x, x* is reduced to a finite spanning set ($R_n = \{x^i(x^*)^j : 0 \leq i, j < n\}$ for B_n), giving dimension bounds. Lower bounds come from the Jordan blocks J_k (Picture 2, right) and the universal property.

$$A_k = \begin{bmatrix} 0 & \alpha_k & 0 & \beta_k \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad J_k = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

$$\alpha_k = \tfrac{k}{k+1},\ \beta_k = \sqrt{1-\alpha_k^2} \qquad J_k \in M_k(C),\ k = 2, \ldots, n$$

Picture 2. Blocks A_k (left) and Jordan block J_k (right)

Discussion & Result

Theorem 1. For every n ≥ 3, U_n is nonunital; hence infinite-dimensional, since every nonzero finite-dimensional C*-algebra is unital.

Proposition 2. In B_n: (i) $x^n = 0$; (ii) B_n is unital with $1 = xx^* + (x^*)^{n-1}x^{n-1}$; (iii) x is a power partial isometry — although this is not imposed in the definition.

Theorem 3. For every n ≥ 2, $B_n \cong M_n(\mathbb{C})$; $B_n = \text{span}(R_n)$.

Theorem 4. For every n ≥ 2, $A_n \cong M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \cdots \oplus M_n(\mathbb{C})$, with $\dim A_n = \Sigma_{k=2}^{n} k^2$. The isomorphism is $x \mapsto J_2 \oplus \cdots \oplus J_n$.

Conclusion

Nilpotency alone does not restore finite-dimensionality: for n ≥ 3 the universal algebra U_n of a nilpotent partial isometry is nonunital and infinite-dimensional, in sharp contrast with $U_2 \cong M_2(\mathbb{C})$.

Finite-dimensionality is recovered by strengthening the relations. Requiring a nilpotent power partial isometry gives $A_n \cong \bigoplus_{k=2}^{n} M_k(\mathbb{C})$, and the explicit polynomial relation cuts this down to a single full matrix algebra $B_n \cong M_n(\mathbb{C})$.

Dimensions. $\dim U_n = \infty$ (n ≥ 3), $\dim A_n = n(n+1)(2n+1)/6 - 1$, $\dim B_n = n^2$.

Thus the chain $U_n \to A_n \to B_n$ passes from infinite-dimensional, through a direct sum of matrix algebras, to $M_n(\mathbb{C})$.

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